Flat families

Recall that an R-module M is <u>flat</u> if for all injections of R-modules $N' \hookrightarrow N$, the map

$$M \otimes_R N' \longrightarrow M \otimes_R N$$

is injective. Since tensoring is always right-exact, tensoring by a flat module is exact.

We showed that localization is exact. i.e. $U^{-1}R$ is a flat R-module.

We will see that flatness is connected to the idea of varieties/schemes, or their corresponding algebras, varying in a family.

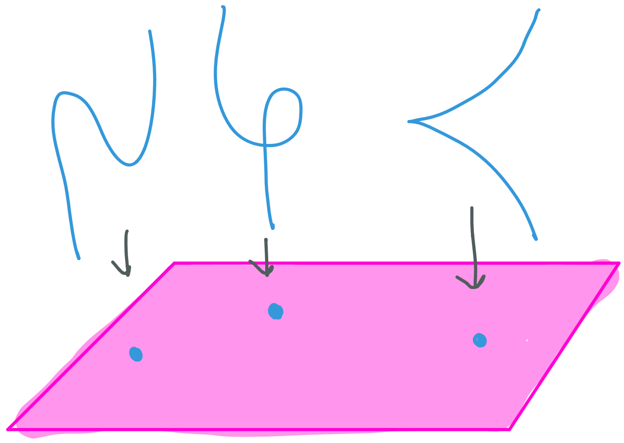
Ex: Plane curves of degree d are described by

$$f = \sum_{i+j \le d} a_{ij} x^i y^j \in k[x,y].$$

Varying coefficients varies the curve. Algebraically, we vary the k algebra $k[x,y]/(f)$.

The a_{ij} parametrize the curves, and we get a map to the "parameter space", where each fiber is the corresponding curve.

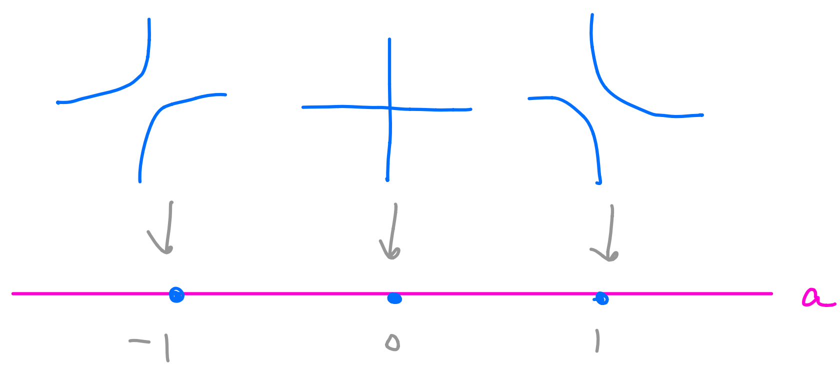
$$\operatorname{Spec} k[a_{ij}] \longrightarrow \operatorname{Spec} k[x,y,a_{ij}]/(f)$$



So why not define a family as a morphism? i.e. each member of the family is a fiber.

This is too general. In curve example, if we vary the parameters of a curve, it changes the geometry in a natural way.

Ex: $xy - a = 0$ gives a "sub-family" of curves, and as $a \to 0$, it deforms from a hyperbola into the union of two lines

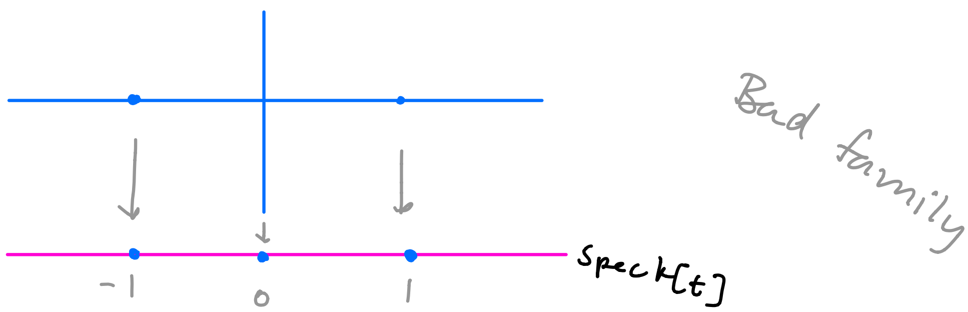


Good family!

However, if we set all coefficients to 0, $f = 0$, so we get the whole plane, or $\operatorname{Spec} k[x,y]$, which doesn't fit nicely in a family of curves.

Similarly, we could have a one-parameter family like this one we've seen before:

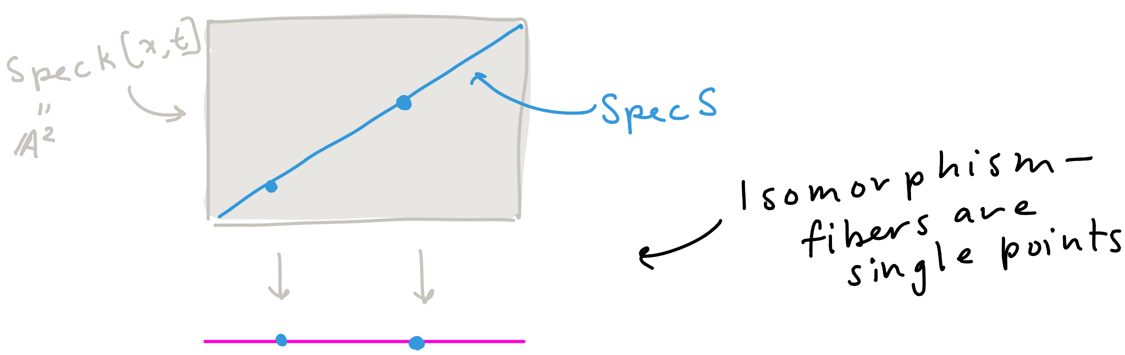
Ex: $\operatorname{Spec} k[x,t]/(xt) \to \operatorname{Spec} k[t]$, $k = \bar{k}$.



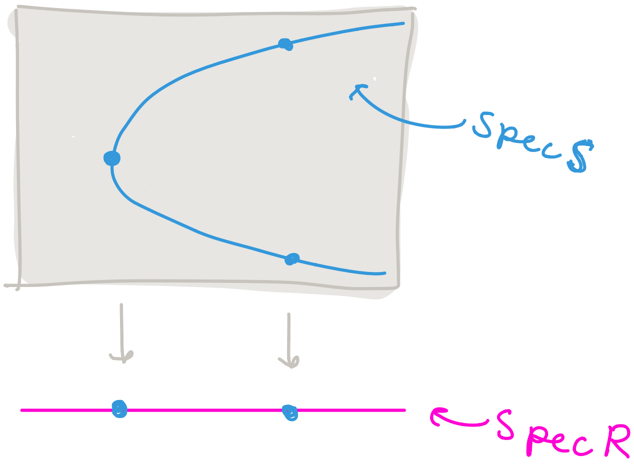
The way we exclude these cases is by requiring the map to be flat. i.e. we want the corresponding map of rings $R \to S$ to make S flat as an R-module.

Ex: Let $R = k[t]$ and $k = \bar{k}$.

1.) $S = R[x]/(x-t)$. Then $S \cong R$ and $R \otimes_R N = N$ for all N, so R is flat. Geometrically:



2.) $S = R[x]/(x^2-t)$



Fiber over $(t-a)$ is

$$\operatorname{Spec}\left(S \otimes_R R/(t-a)\right)$$
$$= \operatorname{Spec}\left(S/(t-a)\right)$$
$$= \operatorname{Spec}\left(k[x]/(x^2-a)\right)$$

= 2 points when $a \neq 0$, 1 point when $a = 0$.

However, notice that $k[x]/(x^2-a) \supseteq (x-\sqrt{a}) \supseteq 0$ for any a, so these all have length 2. Since these are Artinian k-algebras, we can also compute the length as

$$\ell\left(k[x]/(x^2-a)\right) = \dim_k k[x]/(x^2-a) = \dim_k\left(k1 \oplus kx\right) = 2.$$

Moreover, S is a free R-module of rk 2 (x^2-t is monic), so

$$S \otimes_R N \cong R^{\oplus 2} \otimes_R N = N \oplus N$$

for all R-modules N. In particular, tensoring by S preserves injections, so it's flat. (More generally, free modules are always flat.)

We'll soon see that our example above, $k[x,t]/(xt)$, is <u>not</u> a flat $k[t]$-module.

Free resolutions and Tor

When discussing flatness, it's important to know a little homological algebra. Here's a brief intro to Tor:

Let M be an R-module. A <u>free resolution</u> of M is an exact sequence

$$\cdots \to F_1 \to F_0 \to M \to 0$$

where each F_i is a free R-module.

If M is any R-module w/ generating set $\{m_i\}_{i\in J}$, we can construct a free resolution as follows:

Set $F_0 = \bigoplus_{i \in J} R$, and e_i the i^{th} basis vector.

$$\begin{aligned} F_0 &\twoheadrightarrow M \\ e_i &\longmapsto m_i \end{aligned} \quad \text{is surjective.}$$

If $M_0 = \ker(F_0 \twoheadrightarrow M)$, repeat the process to get F_1, and

$$F_1 \to F_0 \to M \to 0.$$

We can continue this to get a (possibly infinite) free resolution of M.

Ex: Let $R = k[x,y]$, $M = R/(x,y^2)$. This is generated as an R-module by 1, so

$$\begin{aligned} R &\to M \to 0 \\ 1 &\longmapsto 1 \end{aligned}$$

which has kernel (x,y^2), so the next step is

$$R^2 \longrightarrow R \longrightarrow M \longrightarrow 0$$
$$(a,b) \longmapsto ax+by^2$$

The kernel is generated by $(y^2, -x)$:

$$0 \to R \longrightarrow R^2 \longrightarrow R \longrightarrow M \to 0$$
$$f \longmapsto (fy^2, -fx)$$

injective

This is a free resolution. Note this is not unique. It requires a choice of generators, and we could just add on superfluous R summands at any point and just send them to 0.

Def: Let M and N be R-modules and

$$\cdots \to F_1 \to F_0 \to M \to 0$$

a free resolution of M. $\mathrm{Tor}_i^R(M,N)$ is the homology at $F_i \otimes N$ of the complex

$$\cdots \to F_{i+1} \otimes N \xrightarrow{\alpha_i} F_i \otimes N \xrightarrow{\alpha_{i-1}} F_{i-1} \otimes N \longrightarrow \cdots \longrightarrow F_0 \otimes N \longrightarrow 0$$

leave out M

i.e. it is $\ker \alpha_{i-1} / \operatorname{im} \alpha_i$

Facts about Tor:

1.) It's well-defined (independent of chosen resolution)

2.) $\mathrm{Tor}_i(M,N) \cong \mathrm{Tor}_i(N,M)$ (i.e. we can compute by finding a resolution of N instead)

3.) $\mathrm{Tor}_0^R(M,N) = \mathrm{coker}(M_1 \otimes N \to M_0 \otimes N) \cong M \otimes N$, since $\otimes$ is right-exact.

4.) If M is free, $0 \to M \to M \to 0$ is a free res.

$$\Rightarrow \mathrm{Tor}_i(M,N) = 0 \quad \forall\, i > 0.$$

5.) $\mathrm{Tor}_i^R(M,N)$ is R-bilinear: mult. on M by $r \in R$ induces mult. by r on $\mathrm{Tor}_i(M,N)$.

6.) If S is a flat R-algebra, then

$$S \otimes_R \mathrm{Tor}_i^R(M,N) = \mathrm{Tor}_i^S(S \otimes_R M, S \otimes_R N) \quad \text{(Exer)}$$

7.) Tor is left derived functor of tensor. That is, if

$$0 \to M' \to M \to M'' \to 0 \quad \text{is exact, then so is}$$

$$\cdots \to \mathrm{Tor}_2(M'',N) \to \mathrm{Tor}_1(M',N) \to \mathrm{Tor}_1(M,N) \to \mathrm{Tor}_1(M'',N) \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0.$$

Ex: Let R be a ring and $x \in R$ a nonzero divisor. Then

$$0 \to R \xrightarrow{\cdot x} R \to R/(x) \to 0$$

is a free resolution. If M is any R-module, then

$$\mathrm{Tor}_i\left(R/(x), M\right) = H_i\left(0 \to M \xrightarrow{\cdot x} M \to 0\right)$$

so $\mathrm{Tor}_0\left(R/(x), M\right) = M/xM$,

$\mathrm{Tor}_1\left(R/(x), M\right) = \{m \in M \mid xm = 0\}$,

and all higher Tor_i are 0.

Tor and flatness

On next HW, we'll see

$$\mathrm{Tor}_1^R(M,N) = 0 \text{ for all } N \iff M \text{ is flat} \iff \mathrm{Tor}_i^R(M,N) = 0 \ \forall\, i > 0.$$

In general, this can be hard to check. The following stronger condition can be easier to check.

Prop: R a ring, M an R-module.

1.) If $I \subseteq R$ is an ideal, $I \otimes_R M \longrightarrow M$ (multiplication map) is an injection $\iff \mathrm{Tor}_1^R(R/I, M) = 0$.

2.) M is flat $\iff$ the condition in 1.) is satisfied for all ideals $I \subseteq R$.

Pf: 1.) The short exact sequence

$0 \to I \to R \to R/I \to 0$ yields the long exact sequence

$$\cdots \to \mathrm{Tor}_1(R,M) \to \mathrm{Tor}_1(R/I, M) \to I \otimes M \to R \otimes M \to R/I \otimes M \to 0.$$

$\mathrm{Tor}_1(R,M) = 0$ (R is free)

So this [$\mathrm{Tor}_1(R/I, M)$] is $0 \iff$ This [$I \otimes M \to R \otimes M$] is injective

2.) ($\Rightarrow$) by def of flatness.

($\Leftarrow$) Assume the equiv. conditions hold for all ideals $I \subseteq R$.

Let $\varphi: N' \hookrightarrow N$ R-modules, and take corresponding map

$$N' \otimes M \longrightarrow N \otimes M$$

Let x be in the kernel. We want to show $x = 0$.

If $x = \sum n_i \otimes m_i$ (finitely many summands), then

$$\varphi(x) = \sum \varphi(n_i) \otimes m_i = 0.$$

This involves only finitely many elements of N, so we can replace N by $\sum R\varphi(n_i)$ and N' by $\sum R n_i$ and assume N is finitely generated, by n generators.

Then N/N' is finitely generated. Let

$$N' = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_t = N, \text{ where } N_{i+1}/N_i \text{ is gen. by one element.}$$

Thus, $R \twoheadrightarrow N_{i+1}/N_i$, so $N_{i+1}/N_i \cong R/I_i$, some ideal I_i.

$$0 \to N_i \to N_{i+1} \longrightarrow N_{i+1}/N_i \to 0 \quad \text{yields}$$

exact sequence $$\cdots \to \underbrace{\mathrm{Tor}_1\Big(\underset{\overset{\shortparallel}{R/I_i}}{N_{i+1}/N_i},\, M\Big)}_{\overset{\shortparallel}{0} \text{ by assumption, so}} \to N_i \otimes M \to N_{i+1} \otimes M \to \cdots$$

is injective for $\forall\, i$.

$\Rightarrow$ $N' \otimes M \to N \otimes M$ is the composition of injections, so it's injective. $\square$

In fact, w/ a little tweaking, one can show the condition only needs to hold for finitely generated ideals I.

From this, we get a nice corollary about modules over a PID. First, a definition:

Def: An R-module M is <u>torsion free</u> if for all $r \in R$ nonzerodivisors (in $\underline{\underline{R}}$), and $m \in M$ nonzero, we have $rm \neq 0$.

Ex: Free modules are always torsion free. So are ideals.

Cor: 1.) If M is a flat R-module, then M is torsion free.

2.) If R is a PID, then M is flat $\Leftrightarrow$ M is torsion free.

Pf: 1.) Let $a \in R$ be a NZD in R. We showed that

$$\mathrm{Tor}_1(R/(a), M) = \{m \in M \mid am = 0\} \underset{\substack{\uparrow \\ \text{flatness}}}{=} 0$$

Thus, a is a NZD on M, so M is torsion free.

2.) ($\Rightarrow$) by 1.)

($\Leftarrow$) Assume M is torsion free over R, a PID, and thus an integral domain.

If $I \subseteq R$ an ideal, then $I = (a)$. WTS $\mathrm{Tor}_1^R(R/(a), M) = 0$.

If $a = 0$, we're done since R is flat. If $a \neq 0$, then $\mathrm{Tor}_1(R/a, M) = \{m \in M \mid am = 0\} = 0$. So by the Prop, M is flat. $\square$

Ex: $k = \bar{k}$, $R = k[t]$, $S = R[x]/(xt)$. S is not torsion free:

$$\underset{\text{in } R}{t}\,\underset{\text{in } S}{x} = 0$$

Thus, S is not flat over R.

Flatness is local! Geometrically, this means we can check for flatness in a neighborhood of each point. Algebraically, it means the following:

Prop: M is a flat R-module $\Leftrightarrow$ M_p is a flat R_p-module for every prime ideal $P \subseteq R$.

Pf: Assume M is flat over R. Let $N' \subseteq N$ be R_p-modules. They are also R-modules, so

$$\begin{tikzcd}
M \otimes_R N' \arrow[r, hook] \arrow[d, phantom, "\cong"] & M \otimes_R N \arrow[d, phantom, "\cong"] \\
M \otimes_R N' \otimes_R R_p \arrow[d, phantom, "\cong"] & M \otimes_R N \otimes R_p \arrow[d, phantom, "\cong"] \\
M_p \otimes_R N' & M_p \otimes_R N
\end{tikzcd}$$

So M_p is a flat R_p-module.

Now assume M is not flat. Then there is some inclusion $N' \hookrightarrow N$ of R-modules such that

$$M \otimes_R N' \longrightarrow M \otimes_R N$$

is not injective, and thus has kernel $K \neq 0$. Thus,

$$0 \to K \to M \otimes_R N' \to M \otimes_R N$$

is exact and there is some prime ideal $P \subseteq R$ s.t. $K_p \neq 0$, so

$$0 \longrightarrow \underset{\substack{\neq \\ 0}}{K_p} \longrightarrow M_p \otimes_R N_p' \longrightarrow M_p \otimes N_p$$

is exact, so is not injective, even though $N_p' \to N_p$ is. Thus M_p is not flat. $\square$

Note that we could replace "prime" w/ "maximal."

Ex: The "problem point" of $M = k[x,t]/(xt)$ as an $R = k[t]$-module is (t). Localizing at (t) we get

$$M_{(t)} = k[x,t]_{(t)} / (tx)_{(t)}$$

which has torsion, since $tx = 0$, so it's not flat.

At any other prime ideal P, t becomes a unit, so

$$M_P = k[x,t]_P / (tx)_P = k[x,t]_P / (x)_P \cong k[t]_P = R_P$$

which is free over R_P and thus flat.